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RAY METHOD OF SOLUTION OF THE ABLATION PROBLEM

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Consideration is given to a three-dimensional problem of heating of a half-space by the Gaussian energy flux with allowance made for the heat-flux inertia. It is assumed that the ablation front is formed at the points at which the temperature is equal to the melting temperature and the Stefan condition is satisfied. The mechanism of removal of mass is not considered. Using the ray method and taking into account three expansion terms, an ablation-surface equation and temperature have been obtained. Account for the inertia of the heat flux leads to the fact that the ablation-front velocity at the initial instant of time is finite. The condition for the onset of ablation of the material at the initial instant of time has been obtained. The solutions for the fronts of ablation and temperature are illustrated by the graphs.

The methods of solution of one-dimensional problems of ablation have been considered in [1]. The ablation of material which takes into account the inertia of the heat flux is described by the generalized heat-conduction law [2] and by the energy conservation law [3]:

$$\tau_0 q_{i,t}^{(1)} + q_i = -\lambda_0 T_{,i}^{(1)} , \quad q_{i,i}^{(1)} + c_p T_{,t}^{(1)} = 0 . \tag{1}$$

Here and in what follows, the subscripts after a comma mean differentiation with respect to the time t or the space Cartesian coordinate x_i, while the superscript in parentheses means the order of a derivative; the condition of summation over double superscripts is taken, with the Latin superscripts taking on values of 1 to 3.

Let us assume that a medium is in an unperturbed state at the initial instant of time $t = 0$. The conditions of conjugation at the ablation boundary have the form

$$g_0 (t) \exp (- d (x_1^2 + x_2^2)) - q_i (x_1, x_2, x_3, t) \nu_i = L\rho V , \quad T (x_1, x_2, x_3, t) = T_L , \tag{2}$$

where $g_0(t) \exp (-d(x_1^2 + x_2^2))$ is the Gaussian energy flux.

To obtain the solution of problem (1)–(2) we pass from the fixed orthogonal Cartesian coordinate system $x_1x_2x_3$ to a moving local coordinate system $y_1y_2\mathbf{n}$ related to a certain surface $\Sigma(y_1, y_2, t)$ which moves with a velocity c and whose coordinates have the form $x_i = x_i(y_1, y_2, t)$ at any instant of time. In the surface coordinate system y_1y_2, we will distinguish the (contravariant) superscripts and (covariant) subscripts for the vector components. On passage, we replace the derivatives with respect to the coordinates x_1 in the differential equations (1) by the derivatives with respect to the local curvilinear coordinates and the normal $\mathbf{n}$. The partial functional derivatives with respect to the coordinate x_1 are related to the derivatives with respect to the surface coordinates y_α by the relations [4]

$$f_{,i}^{(1)} = f_{,n}^{(1)} \nu_i + g^{\alpha\beta} f_{,\alpha}^{(1)} x_{i,\beta} , \quad f_{,t}^{(1)} = - c f_{,n}^{(1)} + \delta f/\delta t . \tag{3}$$

Here and in what follows the subscripts after a comma mean differentiation with respect to the normal $\mathbf{n}$ or the surface coordinate y_α; the Greek superscripts take on values of 1 and 2. The first relation of (3) represents the expansion of the vector components of the gradient $f_{i,t}^{(1)}$ of the function f in three local directions y_1, y_2, and $\mathbf{n}$; this relation is called the geometric condition of consistency of first order [4]. The second relation of (3) is called the kinematic condition of consistency and it relates the time derivative $f_{,t}^{(1)}$ of the function $f(x_i, t)$ to the time derivative $\delta f/\delta t$ of the

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function $f(y_\alpha, t)$. If a certain function f is assigned only on the surface $\Sigma(t)$ as a function of y_α and t, then the δ derivative of this quantity coincides with the partial time derivative. These functions are components of the vectors ν_i and $x_{i,\alpha}$ and of other characteristics of the inner geometry of the surface $\Sigma(t)$. Hereafter, we will interpret the discontinuity surface as a moving layer of thickness $2l$ for $l \to 0$, in which the quantity f changes from f^+ to f^- monotonically and continuously.

Substituting the kinematic and geometric conditions of consistency (3) into system (1), then integrating the system from $-l$ to l and passing to the limit for $l \to 0$, we obtain the system of four homogeneous equations

$$- \tau_0 c \, [q_i] + \lambda_0 \, [T] \, \nu_i = 0 \, , \quad [q_i] \, \nu_i - c_p c \, [T] = 0 \, . \tag{4}$$

Here, the square brackets denote the discontinuity of the quantity on the surface, e.g., $[f] = (f^+ - f^-)\big|_\Sigma$, where the plus and minus signs refer to the value of the function before and after the discontinuity-wave front respectively.

Equating the determinant in system (4) to zero, we obtain the value for the velocity of the heat wave $c_1 = \sqrt{\lambda_0/(\tau_0 c_p)}$. It can be shown that the heat wave represents a vortex-free wave [4], i.e., vortices do not appear after the passage of the heat front.

We introduce the notation $\omega_1 = [q_i]\nu_i\big|_\Sigma$. Then we substitute $c = c_i$ into system (4); as a result, the following relations are satisfied at the heat-wave front:

$$[q_i] = \omega_1 \nu_i \, , \quad [T] = \omega_1/(c_p c_1) \, . \tag{5}$$

We write Eq. (1) in discontinuities:

$$\tau_0 \, [q_{i,t}^{(1)}] + \lambda_0 \, [T_{,i}^{(1)}] = - [q_i] \, , \quad [q_{i,i}^{(1)}] + c_p \, [T_{,t}^{(1)}] = 0 \, . \tag{6}$$

Having substituted the kinematic and geometric consistency conditions for the discontinuities of first order [4] $[f_{,i}^{(1)}] = [f_{,n}^{(1)}]\nu_i + g^{\alpha\beta}[f]_{,\alpha} x_{i,\beta}$, $[f_{,t}^{(1)}] = -c[f_{,n}^{(1)}] + \delta[f]/\delta t$ into Eq. (6), we obtain

$$\begin{gathered} - \tau_0 c \, [q_{i,n}^{(1)}] + \lambda_0 \, [T_{,n}^{(1)}] \, \nu_i = - \tau_0 \delta \, [q_i]/\delta t - \lambda_0 g^{\alpha\beta} \, [T]_{,\alpha} \, x_{i,\beta} - [q_i] \, , \\ [q_{i,n}^{(1)}] \, \nu_i - c_p c \, [T_{,n}^{(1)}] = - g^{\alpha\beta} \, [q_i]_{,\alpha} \, x_{i,\beta} - c_p \delta \, [T]/\delta t \, . \end{gathered} \tag{7}$$

We eliminate the discontinuity $[q_{i,n}^{(1)}]$. To do this we multiply the first equation of (7) by ν_i, sum it over the subscript i, and add up with the second equation multiplied by $\tau_0 c$. As a result, we obtain

$$(\lambda_0 - \tau_0 c_p c^2) \, [T_{,n}^{(1)}] = - \tau_0 \nu_i \delta \, [q_i]/\delta t - [q_i] \, \nu_i - \tau_0 c g^{\alpha\beta} \, [q_i]_{,\alpha} \, x_{i,\beta} - \tau_0 c_p c \delta [T]/\delta t \, .$$

When $c = c_i$, the left-hand side of the equation becomes zero. After the rearrangements and simplifications with account for Eq. (5), we have the equation of attenuation of the discontinuity intensity at the heat-wave front:

$$\delta\omega_1/\delta t + \omega_1/(2\tau_0) = c_1 \Omega \omega_1 \, . \tag{8}$$

Here

$$\Omega = (\Omega_0 - \mathrm{K}_0 c_1 t)/(1 - 2\Omega_0 c_1 t + \mathrm{K}_0 c_1^2 t^2)$$

(Ω_0 and K_0 are the mean and Gaussian curvatures of the discontinuity surface at the initial instant of time [5]). In the general case, the solution of the differential equation (8) has the form

$$\omega_1 = \omega_{01} \, (y_1, y_2) \exp \left(- t/(2\tau_0)\right)/\sqrt{1 - 2\Omega_0 c_1 t + \mathrm{K}_0 c_1^2 t^2} \, ; \tag{9}$$

from Eq. (9) it follows that a change in the heat-wave intensity is determined by two factors: the geometric divergence which is determined by the multiplier $(1-2\Omega_0 c_1 t+\mathrm{K}_0 c_1^2 t^2)^{-1/2}$ and the heat-flux inertia determined by the term exp $(-t/(2\tau_0))$. If the wave surfaces form a set of parallel planes, then the mean and Gaussian curvatures are equal to zero at any instant of time. It is evident from Eq. (9) that the heat-wave intensity attenuates exponentially.

Let us differentiate system (1) m times with respect to the normal **n** and write it in discontinuities:

$$\tau_0\,[q^{(m+1)}_{i,tn\ldots n}]+\lambda_0\,[T^{(m+1)}_{,in\ldots n}]=-\,[q^{(m)}_{i,n\ldots n}]\,,\quad [q^{(m+1)}_{i,in\ldots n}]+c_p\,[T^{(m+1)}_{,tn\ldots n}]=0\,. \tag{10}$$

The kinematic and geometric consistency conditions of $(m+1)$th order [6] for a constant velocity of the discontinuity wave have the form

$$[f^{(m+1)}_{,in\ldots n}]=[f^{(m+1)}_{,n\ldots n}]\,\nu_i+g^{\alpha\beta}\,[f^{(m)}_{,n\ldots n}]_{,\alpha}\,x_{i,\beta}+L^{(m-1)}_{\beta}\,[f]\,x_{i,\beta}\,,$$

$$[f^{(m+1)}_{,tn\ldots n}]=-\,c\,[f^{(m+1)}_{,n\ldots n}]+\delta\,[f^{(m)}_{,n\ldots n}]/\delta t\,. \tag{11}$$

Here

$$L^{(m-1)}_{\beta}\,[f]=\sum_{R=2}^{m+1}C_m^{m-R+1}\,(R-1)\,!\,g^{\alpha_1\beta}\,[f^{(m-R+1)}_{,n\ldots n}]_{,\alpha_R}\,B^{\alpha_R}_{\alpha_1}$$

(C_m^{m-R+1} is the number of combinations and $B^{\alpha_R}_{\alpha_1}=\prod_{N=2}^{R} b_{\alpha_{N-1}}\,\beta_N g^{\alpha_N\beta_N}$).

Substitution of Eq. (11) into Eq. (10) yields

$$-\,\tau_0 c\,[q^{(m+1)}_{i,n\ldots n}]+\lambda_0\,[T^{(m+1)}_{,n\ldots n}]\,\nu_i=-\,\tau_0\delta\,[q^{(m)}_{i,n\ldots n}]/\delta t-\lambda_0 g^{\alpha\beta}\,[T^{(m)}_{,n\ldots n}]_{,\alpha}\,x_{i,\beta}-\lambda_0 L^{(m-1)}_{\beta}\,[T]\,x_{i,\beta}-[q^{(m)}_{i,n\ldots n}]\,,$$

$$[q^{(m+1)}_{i,n\ldots n}]\,\nu_i-c_p c\,[T^{(m+1)}_{,n\ldots n}]=-\,g^{\alpha\beta}\,[q^{(m)}_{i,n\ldots n}]_{,\alpha}\,x_{i,\beta}-L^{(m-1)}_{\beta}\,[q_i]\,x_{i,\beta}-c_p\delta\,[T^{(m)}_{,n\ldots n}]/\delta t\,. \tag{12}$$

We eliminate the discontinuity $[q^{(m+1)}_{i,n\ldots n}]$. To do this we multiply the first equation of (12) by ν_i, sum it over the subscript i, and add up with the second equation multiplied by $\tau_0 c$. As a result, we obtain

$$(\lambda_0-\tau_0 c_p c^2)\,[T^{(m+1)}_{,n\ldots n}]=-\,\tau_0\nu_i\delta\,[q^{(m)}_{i,n\ldots n}]/\delta t-[q^{(m)}_{i,n\ldots n}]\,\nu_i-$$

$$-\,\tau_0 c g^{\alpha\beta}\,[q^{(m)}_{i,n\ldots n}]_{,\alpha}\,x_{i,\beta}-\tau_0 c L^{(m-1)}_{\beta}\,[q_i]\,x_{i,\beta}-\tau_0 c_p c\delta\,[T^{(m)}_{,n\ldots n}]/\delta t\,. \tag{13}$$

When $c=c_1$, the recurrence equation (3) describes the behavior of the discontinuities for the temperature and the heat flux of order m on the discontinuity surface.

For the boundary-value problem (1), the heat wave represents a plane which propagates along the $0x_3$ axis with a velocity c_1. As the curvilinear coordinates on the moving surface we select $y_1=x_1$ and $y_2=x_2$. The Cartesian coordinate of the moving surface is $x_3=c_1 t$. The components of the unit vector of the normal to the plane take on values of $\nu_1=\nu_2=0$ and $\nu_3=1$. Then the components of the fundamental covariant metric tensor of the surface are as follows: $g_{\alpha\beta}=x_{i,\alpha}x_{i,\beta}=\delta_{\alpha\beta}$ and $b_{\alpha\beta}=x_{i,\alpha\beta}\nu_i=0$ [6]. The contravariant components of the metric tensor $g^{\alpha\beta}$ are obtained from the relation $g_{\alpha\sigma}g^{\sigma\beta}=\delta^{\beta}_{\alpha}$ and they take on values of $g^{\alpha\beta}=\delta^{\alpha\beta}$.

Substituting $c=c_1$ for $m=1$, we obtain the equation of attenuation for the intensity of the discontinuity of first order:

$$\delta\omega^{(1)}_{n1}/\delta t+\omega^{(1)}_{n1}/(2\tau_0)=-\,\omega_1/(8\tau_0^2 c_1)-c_1\omega_{1,\alpha\alpha}/2\,. \tag{14}$$

Here $\omega_{n1}^{(1)} = [q_{i,n}^{(1)}]\nu_i|_\Sigma$.

The solution of (14) has the form

$$\omega_{n1}^{(1)} = [\omega_{n01}(y_1, y_2) - (\omega_{01}(y_1, y_2)/(8\tau_0^2 c_1) + c_1\omega_{01,\alpha\alpha}(y_1, y_2)/2)\, t] \exp(-t/(2\tau_0)) . \quad (15)$$

From Eq. (12) for $m = 0$ we obtain

$$[q_{i,n}^{(1)}] = \omega_{n1}^{(1)}\nu_i + \omega_{1,\alpha}x_{i,\alpha} , \quad [T_{,n}^{(1)}] = (\omega_{n1}^{(1)} - \omega_1/(2\tau_0 c_1))/(c_p c_1) . \quad (16)$$

When $m = 2$ and $c = c_1$, from Eq. (13), after the rearrangements and simplifications, we derive the equation of attenuation for the intensity of the discontinuity of second order:

$$\delta\omega_{n1}^{(2)}/\delta t + \omega_{n1}^{(2)}/(2\tau_0) = -\,\omega_{n1}^{(1)}/(8\tau_0^2 c_1) - c_1\omega_{n1,\alpha\alpha}^{(1)}/2 , \quad (17)$$

where $\omega_{n1}^{(2)} = [q_{i,nn}^{(2)}]\nu_i|_\Sigma$.

The solution of Eq. (17) has the form

$$\omega_{n1}^{(2)} = [\omega_{nn01}(y_1, y_2) + A_1 t + B_1 t^2] \exp(-t/(2\tau_0)) . \quad (18)$$

Here

$$A_1 = -\,\omega_{n01}(y_1, y_2)/(8\tau_0^2 c_1) - c_1\omega_{n01,\alpha\alpha}(y_1, y_2)/2 ,$$

$$B_1 = \omega_{01}(y_1, y_2)/(128\tau_0^4 c_1^2) + \omega_{01,\alpha\alpha}(y_1, y_2)/(16\tau_0^2) + c_1^2\omega_{01,\alpha\alpha\beta\beta}(y_1, y_2)/8 .$$

When $m = 1$, from Eq. (12) we obtain

$$[q_{i,nn}^{(2)}] = \omega_{n1}^{(2)}\nu_i + \omega_{n1,\alpha}^{(1)} x_{i,\alpha} ,$$

$$[T_{,nn}^{(2)}] = (\omega_{n1}^{(2)} - \omega_{n1}^{(1)}/(2\tau_0 c_1) + \omega_1/(8\tau_0^2 c_1^2) + \omega_{1,\alpha\alpha}/2)/(c_p c_1) . \quad (19)$$

Solutions for the temperature and the heat flux are represented in the form of the ray series [7]

$$f = (f^+ - [f])|_\Sigma - h\,(f_{,n}^{(1)+} - [f_{,n}^{(1)}])|_\Sigma + \frac{h^2}{2!}(f_{,nn}^{(2)+} - [f_{,nn}^{(2)}])|_\Sigma - \dots ,$$

$$f_{,n\dots n}^{(k)} = \frac{\partial^k f}{\partial x_i \partial x_j \dots \partial x_l}\nu_i\nu_j \dots \nu_l . \quad (20)$$

The velocity of the ablation front is found as

$$V = \sum_{k=0}^{\infty} a_k(y_1, y_2)\, t^k . \quad (21)$$

Integrating Eq. (21) and assuming that the transverse component of the ablation-front gradient is small compared to the longitudinal component, we obtain

$$x_3 = \sum_{k=0}^{\infty} a_k(y_1, y_2)\, t^{k+1}/(k+1) . \quad (22)$$

In the case of propagation of plane discontinuity fronts, the ray series (20) has the form [8]

$$f(y_1, y_2, x_3, t) = -\sum_{k=0}^{\infty} \frac{(x_3 - c_1 t)^k}{k!} [f^{(k)}_{,n\ldots n}]\,\big|_{\Sigma} . \tag{23}$$

In order to construct the solution, it is necessary to find the quantities ω_{01}, ω_{n01}, and ω_{nn01} and a_0, a_1, and a_2. They are determined from conjugation conditions (2). To do this we substitute the ray relations (23) for the temperature and the heat flux with account for Eq. (22) into conjugation conditions (2), setting $t = 0$ and $x_3 = 0$. As the result, we obtain

$$g_0(0)\exp(-d(y_1^2 + y_2^2)) + \omega_{01} = L\rho a_0 , \quad -\omega_{01}/(c_p c_1) = T_L . \tag{24}$$

By solution of this system of equations we find the zero expansion coefficients:

$$\omega_{01} = -T_L c_p c_1 , \quad a_0 = (g_0(0)\exp(-d(y_1^2 + y_2^2)) - T_L c_p c_1)/(L\rho) . \tag{25}$$

The quantity ω_{01} determines the intensity of the discontinuity of the heat wave, while a_0 determines the velocity of the ablation front at the initial instant of time $t = 0$.

It should be noted that in a Neumann classical solution of the melting problem for a half-space with a melting temperature assigned on the surface, the velocity of the boundary between the liquid and solid phases at the initial instant of time takes on an infinite value [3].

From the first relation of Eq. (5) and from Eq. (23) it follows that the heat flux at the initial instant of time has a finite value. The second relation of Eq. (25) shows that the ablation of material begins when the condition $g_0(0)\exp(-d(y_1^2 + y_2^2)) \geq T_L c_p c_1$ is satisfied. From Eq. (21) and from the second relation of Eq. (24) it is evident that the ablation-front velocity at the initial instant of time has a finite value. This fact is explained by the assumption made on the existence of the heat-flux inertia.

The wave surface $\Sigma(y_1, y_2, t)$ off the boundary $x_3 = 0$ is a plane for which the discontinuity intensity ω_1 inside the circle $y_1^2 + y_2^2 \leq \ln(g_0(0)/(T_L c_p c_1))/d$ takes on a value of $\omega_1 = -T_L c_p c_1 \exp(-t/(2\tau_0))$, which is independent of the coordinates y_1 and y_2, whereas outside the circle the discontinuity intensity takes on a value of $\omega_1 = -g_0(0)\exp(-d(y_1^2 + y_2^2)\exp(-t/(2\tau_0))$. The last quantity has been obtained from Eq. (24) for $a_0 = 0$.

To find the coefficients ω_{n01} and a_1 we differentiate Eq. (2) with respect to the time t, setting $t = 0$ and $x_3 = 0$. As a result, we obtain a system of equations whose solution is represented by the first coefficients of the expansion

$$\omega_{n01} = a_0\omega_{01}/(2\tau_0 c_1 (a_0 - c_1)) ,$$

$$a_1 = (g_0'(0)\exp(-d(y_1^2 + y_2^2)) - \omega_{01}/(2\tau_0) + (a_0 - c_1)\,\omega_{n01})/(L\rho) . \tag{26}$$

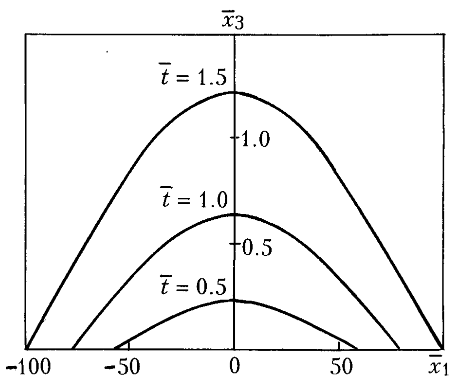
The coefficient ω_{n01} is the value of $\omega_{n1}^{(1)} = [q^{(1)}_{i,n}]\nu_i\big|_{\Sigma}$ at the initial instant of time ($t = 0$). From Eq. (21) it follows that a_1 determines the acceleration of the points of the ablation front at the initial instant of time, which depends on the coordinates selected on the moving surface.

The coefficients ω_{nn01} and a_2 are determined in a similar manner. We differentiate Eq. (2) with respect to the time t twice and set $t = 0$ and $x_3 = 0$. Solution of the system of equations yields

$$\omega_{nn01} = 1/(a_0 - c_1)^2 \Big\{\omega_{01}/(4\tau_0^2) + a_1\omega_{n01} - a_1\omega_{01}/(2\tau_0 c_1) +$$

$$+ 2(a_0 - c_1)(-\omega_{n01}/(2\tau_0) + \omega_{01}/(8\tau_0^2 c_1))\Big\} + \omega_{n01}/(2\tau_0 c_1) - \omega_{01}/(8\tau_0^2 c_1^2) ,$$

$$a_2 = (g_0''(0)\exp(-d(y_1^2 + y_2^2)) + \omega_{01}/(4\tau_0^2) + a_1\omega_{n01} +$$



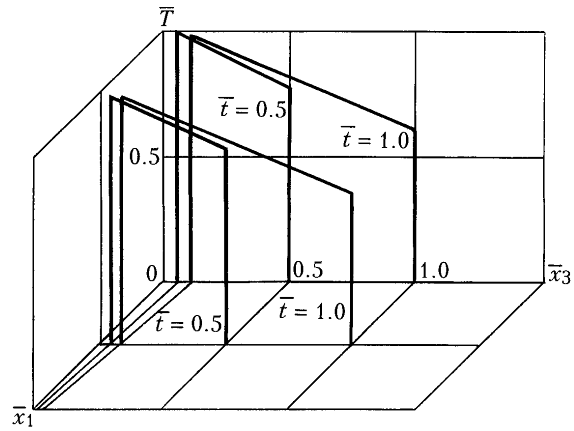


Fig. 1. Position of the ablation fronts at the instants of time $\bar{t}$ = 0.5, 1.0, and 1.5.

Fig. 2. Temperature for $\bar{x}_1$ = 0 and 50 and for $\bar{x}_2$ = 0. (27)

$$+ 2\,(a_0 - c_1)\,(-\,\omega_{n01}/(2\tau_0) - \omega_{01}/(8\tau_0^2 c_1)) + (a_0 - c_1)^2\,\omega_{nn01})/(2L\rho)\;.$$

The coefficients ω_{nn01} and a_2 represent the initial values of $\omega_{n1}^{(2)} = [q_{i,nn}^{(1)}]\nu_i\,|_{\Sigma}$ and half the change in the acceleration of the ablation-front points with time respectively.

By way of illustration of the solutions obtained we select, as a material, aluminum with the following thermophysical properties: ρ = 2700 kg/m^3, λ_0 = 209.3 W/(m·deg), $\tau_0 = 10^{-11}$ sec, c_p = 2.376 kJ/(m^3·deg), $T_L = 658^{\circ}$C, and L = 358 kJ/kg.

In dimensionless form, the differential equations and the conjugation conditions will be written as follows:

$$\bar{q}_{i,t}^{(1)} + \bar{q}_i = -\,\bar{T}_{,i}^{(1)}\,,\quad \bar{q}_{i,i}^{(1)} + \bar{T}_{,t}^{(1)} = 0\,,$$

$$\bar{g}_0\,(\bar{t})\,\exp\,(-\,\bar{d}\,(\bar{x}_1^2 + \bar{x}_2^2)) - q_i\,(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{t})\,\nu_i = \xi \bar{V}\,, \tag{28}$$

$$\bar{T}_L\,(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{t}) = 1\;.$$

From the system of equations (28) given above it is evident that solutions can always be expressed by the coordinates $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, the time $\bar{t}$, and the parameter ξ and that they also depend on the value of the heat flux $\bar{g}_0(t)$ exp $(-\bar{d}(\bar{x}_1^2 + \bar{x}_2^2\,)$.

Since the coordinates on the surface $\Sigma(y_1, y_2, t)$ are equal to $y_1 = x_1$ and $y_2 = x_2$, formula (22) describes the ablation surface at fixed values of time t. The coefficients a_0, a_1, and a_2 are determined from the second relations of Eqs. (25)–(27), which depend on the quantity $x_1^2 + x_2^2$. It follows that the surface of ablation represents the surface of revolution. Figure 1 depicts the position of the ablation fronts at different instants of time $\bar{t}$ = 0.5, 1.0, and 1.5 for $\bar{x}_2$ = 0 with account for three terms of the expansion of series (22) for the values of the heat flux $\bar{g}_0$ = 1.2 and $\bar{d}$ = 10^{-4}. The units of distances and time are equal to $2.97 \cdot 10^{-8}$ m and 10^{-11} sec respectively.

Figure 2 plots the temperatures at the instants of time $\bar{t}$ = 0.5 and 1.0 as functions of $\bar{x}_3$ for $\bar{x}_1$ = 0 and 50 and for $\bar{x}_2$ = 0. The value of the heat load is $\bar{g}_0$ = 5 and ξ = 50. The unit of temperature is equal to 658°C. The temperature is determined in the following manner. First the values of ω_{01}, ω_{n01}, and ω_{nn01} from relations (25)–(27) are substituted into the expressions of ω_1, $\omega_{n1}^{(1)}$, and $\omega_{n1}^{(2)}$ determined by Eqs. (9), (15), and (18). Knowledge of these quantities makes it possible to find the discontinuities for the temperature [T] and their derivatives with respect to the normal $[T_{,n}^{(1)}]$ and $[T_{,nn}^{(2)}]$ at the wave-surface front from the second relations of Eqs. (5), (16), and (19). The final solution for the temperature is written in the form of the ray series (23). The graphs of the temperature, intersecting the plane $0\bar{x}_1\bar{x}_3$, form curvilinear trapezoids whose bases are the leading front propagating with a dimensionless heat-wave velocity $\bar{c}_1$ = 0 and the temperature at the ablation front. The value of the jump at the leading front of the temperature is determined by the second relation of Eq. (5). As has been noted earlier, this front attenuates exponentially with time. It is seen from the first expression of (25) that at a fixed time $\bar{t}$ the value of the temperature jump does not change during the propagation of the plane fronts. If we use the Fourier classical model of propagation of heat, this front will be absent. The ablation fronts are depicted in the plane $0\bar{x}_1\bar{x}_3$ at the instants of time $\bar{t} = 0.5$ and 1.0. The values of the temperatures at the ablation front for $\bar{x}_1 = 0$ at the instants of time $\bar{t} = 0.5$ and 1.0 are equal to 0.995 and 0.986 respectively, while for $\bar{x}_1 = 50$ they are equal to 0.996 and 0.991 respectively. The difference of the above values from the dimensionless temperature of ablation $\bar{T}_L = 1$ amounts to less than 2%.

NOTATION

$a_k(y_1$ and $y_2)$, coefficients of the series for the ablation-front velocity; $b_{\alpha\beta}$, coefficients of the second fundamental quadratic form of the surface, m^{-1}; c and c_1, velocities of the discontinuity-wave front and of the heat wave, m/sec; c_p, specific heat at constant pressure, J/(m^3·deg); d and $\bar{d} = d\lambda_0\tau_0/c_p$, constants; $g_0(t)$, time function; $g_0'(0)$ and $g_0''(0)$, first and second time derivatives of the function $g_0(t)$ at $t = 0$; $g^{\alpha\beta}$, metric surface tensor; $\mathbf{n}$, normal to the surface; h, distance along the normal behind the front of a strong-discontinuity surface, m; L, latent heat of melting, J/kg; $2l$, thickness of the layer, m; q_i, components of the heat-flux vector, W/m^2; $\bar{q}_i = q_i\sqrt{\tau_0/(\lambda_0 c_p)}/T_L$, dimensionless components of the heat-flux vector; t, time, sec; $\bar{t} = t/\tau_0$, dimensionless time; T, temperature, $^{\circ}$C; $\bar{T} = T/T_L$, dimensionless temperature; T_L, melting temperature, $^{\circ}$C; V and $\bar{V}$, velocity and dimensionless velocity of the ablation front, m/sec; x_i, Cartesian coordinates, m; $\bar{x}_i = \sqrt{c_p/(\lambda_0\tau_0)}\, x_i$, dimensionless Cartesian coordinates; y_1 and y_2, curvilinear coordinates on the discontinuity surface, m; δ_α^β and $\delta^{\alpha\beta}$, Kronecker symbols; $\delta/\delta t$, δ derivative with respect to time [4]; K_0, Gaussian curvature of the discontinuity surface at $t = 0$, m^{-2}; λ_0, thermal-conductivity coefficient, W/(m·deg); ν_i, components of the unit vector of the normal to the surface; $\xi = L\rho/(c_pT_L)$, reduced latent heat; ρ, density, kg/m^3; $\Sigma(t)$, discontinuity surface; τ_0, relaxation time of the heat flux, sec; Ω and Ω_0, mean curvatures of the wave surface at any instant of time and at $t = 0$ [4], m^{-1}; ω_1, intensity of the discontinuity at the heat-wave front, W/m^2; $\omega_{n1}^{(1)} = [q_{i,n}^{(1)}]\nu_i\big|_\Sigma$, W/m^3; $\omega_{n1}^{(2)} = [q_{i,nn}^{(2)}]\nu_i\big|_\Sigma$, W/m^4. Subscripts: comma, differentiation; small Latin letters, Cartesian orthogonal coordinate system; n, normal; small Greek letters, curvilinear surface coordinate system.

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